Announcements

1) HW #3 due Thursday

2) Lab, exam 1 E.C.
due today

Example 1: Show

$$\lim_{(x,y)\to(0,4)} \frac{x^2(y-4)^2}{x^4+(y-4)^4}$$

does not exist.

idea: pick two curves, either $y=f(x)$, $x=f(y)$, or more general that go through $(0,4)$. Plug them in to the equation, take a 1 variable limit, get different answers.

Choose $y=4$

Then the equation becomes

$$\frac{x^2(4-4)^2}{x^4+(4-4)^4}$$

$$= \frac{0}{x^4} = \boxed{0}$$

Since $x \neq 0$.

Choose $x=0$

We get

$$\frac{0^2(y-4)^2}{0^4+(y-4)^4}$$

$$=\frac{0}{(y-4)^4}=0$$

Since $y \neq 4$ bad!

This is the same number we just came up with.

Choose a line with slope 1

Need to pass through the point $(0,4)$! Use point-slope form:

$$1 \cdot (x-0) = y-4,$$

so $\boxed{y = x+4}$

plug into equation.

Plugging $y=x+4$ into

$$\frac{x^2(y-4)^2}{x^4+(y-4)^4},$$ so we get

$$\frac{x^2(x+\cancel{4-4})^2}{x^4+(x+\cancel{4-4})^4}$$

$$= \frac{x^4}{2x^4} = \boxed{\frac{1}{2}} \neq 0,$$

So the limit does not exist!

Example 2.

Show

$$\lim_{(x,y)\to(-3,1)} \frac{(x+3)(y-1)^2}{(x+3)^2+(y-1)^4}$$

does not exist

Choose $x=-3$

$$\frac{(0)(y-1)^2}{0^2+(y-1)^4} = \frac{0}{(y-1)^4}$$

$$= \boxed{0}$$

since $y \neq 1$.

Choose $y=1$

$$\frac{(x+3)(0)^2}{(x+3)^2+0^4} = \frac{0}{(x+3)^2}$$

$$= 0$$

Since $x \neq -3$ bad!

Choose a line with slope 1, passing through $(-3,1)$

$\boxed{(x+3) = y-1}$, so

$y = x+4$ plug in!

Plug in $y-1$ for $x+3$ into

$$\frac{(x+3)(y-1)^2}{(x+3)^2+(y-1)^4}$$

Plug in $y-1$ for $x+3$ into

$$\frac{(x+3)(y-1)^2}{(x+3)^2+(y-1)^4}$$

$$\to \frac{(y-1)(y-1)^2}{(y-1)^2+(y-1)^4}$$

$$= \frac{(y-1)\cancel{(y-1)^2}}{\cancel{(y-1)^2}\left(1+(y-1)^2\right)}$$

$$= \frac{(y-1)}{1+(y-1)^2} \to 0 \text{ as } y \to 1$$

bad!

Idea: factor out either

$(y-1)^4$ or $(x+3)^2$

from the denominator.

Original equation:

$$\frac{(x+3)(y-1)^2}{(x+3)^2+(y-1)^4}$$

Try $y-1=\sqrt{x+3}$.

$(y-1)^2 = x+3,$

$(y-1)^4 = (x+3)^2.$

Plug in:

$$\frac{(x+3)(x+3)}{(x+3)^2+(x+3)^2}$$

$$= \frac{\cancel{(x+3)^2}}{2\cancel{(x+3)^2}} = \frac{1}{2} \neq 0,$$

So limit does not exist!

Moral: look at ratios between the powers of x and y to choose your curves!

Remember: Your curves must pass through the point (a,b) in question! Start with lines, work up from there.

Recall Terrifying Definition

$f: D \to \mathbb{R}$ where D is in $\mathbb{R}^2$

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \text{ if}$$

for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\boxed{|f(x,y) - L| < \varepsilon} \text{ when}$$

$$\boxed{0 < \|(x,y) - (a,b)\| < \delta}$$

You don't have to do circles!

Example 4: Show

$$\lim_{(x,y)\to(2,3)} (x+y) = 5$$

Choose $\varepsilon > 0$. We

want

$$\boxed{|x+y-5| < \varepsilon}$$ when

$$0 < \|(x,y)-(2,3)\| < \delta$$

for some δ.

$$|x+y-5|$$

$$= |x+y-2-3|$$

$$= |(x-2)+(y-3)|$$

$$\leq |x-2|+|y-3|$$

Want: Some number k such that

$$|x-2|+|y-3| \leq k \, \|(x,y)-(2,3)\|$$

$$(k>0)$$

Compare

$$\left(|x-2|+|y-3|\right)^2 \text{ and}$$

$$\|(x,y)-(2,3)\|^2 = (x-2)^2+(y-3)^2$$

$$= (x-2)^2 + 2|x-2||y-3|$$

$$+ (y-3)^2$$

$$\geq \|(x,y)-(2,3)\|^2$$

Can't choose $k=1$!

Will $k=2$ work?

We'd have

$$|x-2|+|y-3| \overset{?}{\leq} 2\,\|(x,y)-(2,3)\|.$$

Squaring, we'd get

$$\cancel{(x-2)^2} + 2|x-2||y-3| + \cancel{(y-3)^2}$$

$$\overset{?}{\leq} \underset{3}{\cancel{4}}(x-2)^2 + \underset{3}{\cancel{4}}(y-3)^2$$

So: Is $2|x-2||y-3|$

$$\leq 3(x-2)^2 + 3(y-3)^2 \text{ ?}$$

This is true since

$$2|x-2||y-3|$$

$$\leq 2 \text{ maximum}\{(x-2)^2, (y-3)^2\}$$

$$\leq 3 \text{ maximum}\{(x-2)^2, (y-3)^2\}$$

$$\leq 3(x-2)^2+3(y-3)^2$$ ✓

So $k=2$!

We choose $\delta = \frac{\varepsilon}{2}$.

Then if $0 \leq \|(x,y)-(2,3)\| < \frac{\varepsilon}{2}$,

$$
\begin{aligned}
|x+y-5| &= |x-2+y-3| \\
&\leq |x-2| + |y-3| \\
&\leq 2\|(x,y)-(2,3)\| \\
&< 2(\varepsilon/2) = \varepsilon
\end{aligned}
$$

Done!

Using the following facts:

$$\lim_{(x,y)\to(a,b)} x = a$$ and

$$\lim_{(x,y)\to(a,b)} y = b$$, we

can show that the limit as $(x,y)\to(a,b)$ of any polynomial in x and y exists, using the limit laws.

Example: $p(x,y) = 5x^2y + 10xy^5$

Limit Properties

Suppose $\lim_{(x,y)\to(a,b)} f(x,y) = L$

and $\lim_{(x,y)\to(a,b)} g(x,y) = M$,

where M and L are real numbers.

1) $\lim_{(x,y)\to(a,b)} \left(f(x,y)+g(x,y)\right)$

$= M + L$

2) $\lim_{(x,y)\to(a,b)} \left(f(x,y) g(x,y) \right) = LM$

3) $\lim_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$

except when $M = 0$.

4) Suppose $h: \mathbb{R} \to \mathbb{R}$

and $\lim_{x\to L} h(x) = K$.

Then $\lim_{(x,y)\to(a,b)} h(f(x,y))$

$= \lim_{x\to L} h(x) = K$

The Squeeze Theorem

Suppose $f(x,y) \leq h(x,y) \leq g(x,y)$ on some disk of radius r about (a,b). Then if

$$\boxed{\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} g(x,y) = L}$$

(L a real number), then

$$\lim_{(x,y)\to(a,b)} h(x,y) = L$$

Example 4:

Show

$$\lim_{(x,y)\to(0,0)} \left(x^6 y^2 \sin\left(\frac{16x}{x^2+y^2} \right) \right) = 0.$$

For any real number θ,

$$-1 \leq \sin(\theta) \leq 1$$

With $\theta = \dfrac{16x}{x^2+y^2}$,

$$-1 \le \sin\left(\frac{16x}{x^2+y^2}\right) \le 1 .$$

Multiplying through by x^6y^2,

$$-x^6y^2 \le x^6y^2\sin\left(\frac{16x}{x^2+y^2}\right) \le x^6y^2$$

$$\lim_{(x,y)\to(0,0)} x^6y^2 = 0 = \lim_{(x,y)\to(0,0)} (-x^6y^2),$$

and so by the squeeze theorem,

$$\boxed{\lim_{(x,y)\to(0,0)} x^6 y^2 \sin\left(\frac{16x}{x^2+y^2}\right) = 0}$$